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THE CONTROLLABILITY OF A NON-LINEAR SYSTEM IN A CLASS OF GENERALIZED CONTROLS*

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When studying controllability in non-linear control systems

$$x^{\cdot} = f(t, x, u), \; x \in \mathbb{R}^n, \; u \in \Omega, \; t \in [\tau, \tau + \sigma] \; (\sigma > 0) \tag{0.1}$$

where Ω is a compact subset of $\mathbb{R}^m$, it is generally assumed that

$$0 \in \operatorname{ri}(\operatorname{conv}\Omega) \tag{0.2}$$

and that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ the set $P(t, x) \doteq \{f(t, x, u), u \in \Omega\}$ is a convex, and that τ is fixed. In this paper condition (0.2) will be replaced by the weaker condition $0 \in \operatorname{conv}\Omega$ and the controllability of system (0.1) will be considered with respect to a class of generalized controls. This makes it possible to drop the assumption that the set $P(t, x)$ is convex; τ will not be fixed. Answers will be given to the questions posed in /1/.

1. Let $\mathbb{R}^n$ be a real n-dimensional space, $|x|$ the norm of an element $x \in \mathbb{R}^n$, $B_r[0] \doteq \{x \in \mathbb{R}^n : |x| \leqslant r\}$, $B_r(0) \doteq \operatorname{int} B_r[0]$, $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$, in which the norm of an element A is defined by $|A| \doteq \sup_{x \neq 0} |Ax| / |x|$; $\operatorname{comp}(\mathbb{R}^n)$, $\operatorname{conv}(\mathbb{R}^n)$ denote the spaces of non-empty compact and convex compact subsets of $\mathbb{R}^n$, respectively, each with the Hausdorff metric $\operatorname{dist}(\cdot, \cdot)$.

Let $(X^{(n, m)}, \rho_1^{(l)})$ $(l > 0)$ denote the metric space of locally summable maps $A : \mathbb{R} \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, where

$$\rho_1^{(l)}(A, B) \doteq \sup_t \frac{1}{l} \int_t^{t+l} |A(s) - B(s)| \, ds, \; t \in \mathbb{R}, \; A, B \in X^{(n, m)}$$

$(Y^{(n)}, \rho_2^{(l)})$ is the metric space of all many-valued maps $V : \mathbb{R} \to \operatorname{conv}(\mathbb{R}^n)$ such that the map $t \to |V(t)| \doteq \operatorname{dist}(\{0\}, V(t))$ is measurable (measurability will be understood throughout in the Lebesgue sense), $\operatorname{esssup}_{t \in \mathbb{R}} |V(t)| \leqslant k_V < \infty$ and

$$\rho_2^{(l)}(V, W) \doteq \sup_t \frac{1}{l} \int_t^{t+l} \operatorname{dist}(V(s)), W(s)) \, ds, \; t \in \mathbb{R}, \; V, W \in Y^{(n)}$$

Fix $V \in Y^{(n)}$ and a segment $T \subset \mathbb{R}$. Let $U \doteq B_{k_V}[0]$, $L_T \doteq L(T, U; \mathbb{R}^n)$ be the normed space of all functions $\varphi : T \times U \to \mathbb{R}^n$ such that the map $t \to \varphi(t, u)$ is measurable, $\varphi(t, \cdot) \in C(U) \doteq C(U, \mathbb{R}^n)$, there exists a summable function $\psi : T \to \mathbb{R}$ such that for almost every (a.e.) $|\varphi(t, \cdot)|_{\max} \doteq \max_{u \in U} |\varphi(t, u)| \leqslant \Psi(t)$. Let $\operatorname{frm}(U)$ denote the linear space of Radon measures in $\mathbb{R}^n$ with support in U, and $\operatorname{rpm}(U)$ the subset of $\operatorname{frm}(U)$ consisting of all regular probability measures.

Furthermore, let N_T be the set of measurable maps $\mu : T \to \operatorname{frm}(U)$ such that $\operatorname{esssup}_{t \in T} |\mu|(t)|(U) < \infty$ ($|\mu(t)|(U)$ is the variation of the measure $\mu(t)$). It can be shown /2/, that N_T is algebraically isomorphic to the dual space of $L(T, U; \mathbb{R}^1)$ and one can define in N_T a weak norm $\|\cdot\|_w$ /2/, such that the space $(N_T, \|\cdot\|_w)$ is separable and its subset $U(N_T) \doteq \{\mu \in N_T : |\mu(t)|(U) \leqslant 1\}$ is compact. Moreover, if $\mu_j \in U(N_T)$, $j = 0, 1, \ldots$, then $\lim_{j \to \infty} \|\mu_j - \mu_0\|_w = 0$ (we write $\mu_j \to \mu_0$ as $j \to \infty$) if and only if, for any $\varphi \in L_T$,

$$\lim_{j \to \infty} \int_T \langle \mu_j(t), \varphi(t, u) \rangle \, dt = \int_T \langle \mu_0(t), \varphi(t, u) \rangle \, dt$$

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where

$$\int_T \langle \mu_j(t), \varphi(t,u)\rangle\, dt \doteq \int_T \Big(\int_U \varphi(t,u)\,\mu_j(t)(du)\Big)\, dt,\ j=0,1,\ldots \tag{1.1}$$

Let $I(V)$ denote the set of measurable sections of a map $V \in Y^{(n)}$, $M(V) \doteq \{\mu \in N_{\mathbb{R}} : \mu(t) \in \mathrm{rpm}(V(t))$ for a.e. $t \in \mathbb{R}\}$, $M^1(V) \doteq \{\mu \in M(V) : \mu(t) = \delta_{u(t)}$ for a.e. $t \in \mathbb{R}$ and some $u \in I(V)\}$, where $\delta_{u(t)}$ is the Dirac measure at the point $u(t)$, and let $I_{\tau,\sigma}(V)$, $M_{\tau,\sigma}(V)$, $M^1_{\tau,\sigma}(V)$ $(\tau \geqslant 0,\ \sigma \geqslant 0)$ denote the restrictions of the sets $I(V)$, $M(V)$, $M^1(V)$ to the interval $[\tau, \tau+\sigma]$ on the t axis. Identifying each function $u \in I(V)$ with $\delta_{u(t)}$, we may view $I(V)$ as embedded in $M(V)$. Henceforth we shall call $M^1(V)$ the set of (ordinary) admissible controls and $M(V)$ the set of generalized controls.

We shall be concerned with the non-linear control system

$$x^{\cdot} = A(t)x + u + g(t, x, u),\ x \in \mathbb{R}^n,\ u \in V(t) \tag{1.2}$$

where $A \in X^{(n,n)}$, $V \in Y^{(n)}$ and

$$0 \in V(t),\ t \in \mathbb{R} \tag{1.3}$$

Note that condition (1.3) does not exclude the possibility that $0 \in \partial V(t)$ for some or all t. It is assumed that the function $g : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n$ satisfies the following conditions:

1) for fixed $(x, u) \in \mathbb{R}^n \times U$, the map $t \to g(t, x, u)$ is measurable, $g(t, \cdot, \cdot) \in C(\mathbb{R}^n \times U)$, $t \in \mathbb{R}$, and for any compact set $K \subset \mathbb{R}^n$ there exists a function $\varkappa \in X^{(1,1)}$ such that $\max\{|g(t, x, u)|, (x, u) \in K \times U\} \leqslant \varkappa(t)$ for a.e. $t \in \mathbb{R}$;

2) there exist functions $a, b \in X^{(1,1)}$ and constants $\alpha, \beta > 0$ such that for some $\gamma > 0$ $|g(t, x, u)| \leqslant a(t)|x|^{\alpha} + b(t)|u|^{\beta}$ for all $(t, x, u) \in \mathbb{R} \times B_\gamma[0] \times (U \cap B_\gamma[0])$.

We shall also assume that system (1.2) has the right uniqueness property: for any $x_0 \in \mathbb{R}^n$ and any $u_0 \in I(V)$ the Cauchy problem

$$x^{\cdot} = A(t)x + u_0(t) + g(t, x, u_0(t)),\quad x(\tau) = x_0\ (\tau \geqslant 0)$$

has a solution which is unique over the right maximum interval of its existence.

Definition 1.1. A convex control system corresponding to (1.2) is a system

$$x^{\cdot} = A(t)x + \langle \mu(t),\ u + g(t, x, u)\rangle,\quad \mu \in M(V) \tag{1.4}$$

Clearly, if $\mu \in M^1(V)$, i.e., $\mu(t) = \delta_{v(t)}$, $v \in I(V)$, then system (1.4) yields system (1.2) with $u = v(t)$.

Definition 1.2. A state $x_0 \in \mathbb{R}^n$ is said to be controllable in the interval $[\tau, \tau+\sigma]$ $(\sigma > 0)$ if there exists a generalized control $\nu \in M_{\tau,\sigma}(V)$, such that system (1.4) with $\mu(t) = \nu(t)$ has a solution $y(t)$ satisfying the conditions $y(\tau) = x_0$, $y(\tau+\sigma) = 0$. The set of all controllable states of system (1.4) in the interval $[\tau, \tau+\sigma]$ is called the controllable set of the system in that interval.

Definition 1.3. System (1.4) is said to be uniformly locally controllable if there exist numbers $\varepsilon, \sigma > 0$ such that for any $\tau \geqslant 0$ the controllable set of system (1.4) in the interval $[\tau, \tau+\sigma]$ contains the sphere $B_\varepsilon[0]$.

Together with system (1.4), let us consider the system

$$x^{\cdot} = A(t)x + \langle \mu(t), u\rangle,\ \mu \in M(V) \tag{1.5}$$

It can be shown that, if $D(\tau, \sigma, V)$ denotes the controllable set of system (1.5) in $[\tau, \tau+\sigma]$, then

$$D(\tau, \sigma, V) = \Big\{-\int_\tau^{\tau+\sigma} \langle \mu(t), \Phi(\tau, t)u\rangle\, dt, \mu \in M_{\tau,\sigma}(V)\Big\}$$

where $\Phi(\cdot,\cdot)$ is the Cauchy operator of the system $x^{\cdot} = A(t)x$. It follows from the equality $\{\langle \mu(t), \Phi(\tau, t)u\rangle, \mu \in M_{\tau,\sigma}(V)\} = \mathrm{conv}\{\Phi(\tau, t)V(t)\}$, $t \in [\tau, \tau+\sigma]$ and the condition $V(t) \in \mathrm{conv}(\mathbb{R}^n)$ that

$$D(\tau, \sigma, V) = -\int_\tau^{\tau+\sigma} \Phi(\tau, t)V(t)\, dt$$

The integral on the right of this equality is understood as a Lyapunov integral /3, p. 239/. Hence system (1.5) is uniformly locally controllable (ULC) in the class of generalized controls if and only if the linear system

$$x^{\cdot} = A(t)x + u, \ u \in V(t) \tag{1.6}$$

is ULC.

Lemma 1.1. Let system (1.5) be equipolarly locally controllable, and let $\varepsilon, \sigma > 0$ be the numbers appearing in the definition of ULC for the system.

Assume, moreover, that the function $g(t, x, u)$ satisfies conditions 1, 2 and that at least one of the following conditions holds:

$$s_{a+b} \leqslant \lambda, \quad \text{if} \quad \alpha = \beta = 1 \tag{1.7}$$

$$s_a \leqslant \lambda, \ \text{if} \quad \alpha = 1, \ \beta > 1$$

$$s_b \leqslant \lambda, \ \text{if} \quad \alpha > 1, \ \beta = 1$$

$$\left(c \doteq \rho_1^{(1)}(A, O), \quad \lambda \doteq \frac{\varepsilon}{6\sigma} e^{-4c\sigma}, \quad s_q \doteq \sup_{\tau \geqslant 0} \int_\tau^{\tau+\sigma} q(s)\, ds\right)$$

Then there exists $\delta \in (0, \gamma]$ such that for any function $y \in C(R, B_\delta[0])$ the system

$$x^{\cdot} = A(t)x + \langle \mu(t), u + g(t, y(t), u)\rangle, \ \mu \in M(\delta V) \tag{1.8}$$

is ULC and for any $\tau \geqslant 0$ and

$$x_0 \in B_{\varepsilon\gamma_1/2}[0], \ \gamma_1 \doteq e^{-2c\sigma}\delta/(3\sigma) \tag{1.9}$$

there exists a generalized control $\nu \in M_{\tau,\sigma}(\delta V)$ such that if $\mu(t) = \nu(t)$ system (1.8) has a solution $z(t)$ such that

$$z(\tau) = x_0, \ z(\tau + \sigma) = 0 \quad \text{and} \quad z(t) \in B_\delta[0] \quad \text{for} \quad t \in [\tau, \tau + \sigma].$$

Proof. The constant $\delta > 0$ is chosen in the same way as the analogous constant in the case of ordinary controls (see Lemma 2.1 in /4/). A direct check will convince the reader that

$$D_\tau^1(\sigma, y, \delta V) \doteq \left\{-\int_\tau^{\tau+\sigma} \langle \mu(t), \Phi(\tau, t)(u + g(t, y(t), u)\rangle\, dt, \quad \mu \in M_{\tau,\sigma}(\delta V)\right\}$$
$$y \in C(T, B_\delta[0]), \quad T \doteq [\tau, \tau + \sigma]$$

is the controllable set for system (1.8) in the interval $[\tau, \tau + \sigma]$. Now, noting that

$$e^{2c\sigma} \geqslant \max_{t \in T} |\Phi(\tau, t)|, \quad \mu(t) \in \operatorname{rpm}(\delta V(t))$$

for a.e. $t \in T$, one completes the proof by analogy with Lemma 2.1 of /4/.

2. *Theorem 2.1.* Let system (1.5) be ULC (which is true, as noted above, if and only if system (1.6) is ULC), and let $\varepsilon, \sigma > 0$ be the numbers appearing in the definition of ULC for this system. Assume that the function $g(t, x, u)$ satisfies conditions 1, 2. Then, if one of conditions (1.7) is satisfied, system (1.4) is ULC.

Proof. If one of conditions (1.7) is satisfied, then by Lemma 1.1 there exists $\delta \in (0, \gamma]$ such that system (1.8) is ULC. Choose any $\tau \geqslant 0, x_0$ satisfying condition (1.9) and a function $y_1 \in C(T, B_\delta[0])$, where $T \doteq [\tau, \tau + \sigma]$. Again by Lemma 1.1, there exists a generalized control $\mu_1 \in M_{\tau,\sigma}(\delta V)$, such that if $\mu(t) = \mu_1(t)$ and $y(t) = y_1(t)$ system (1.8) has a solution $y_2(t)$ such that $y_2(t) \in B_\delta[0]$, $t \in T$, and $y_2(\tau) = x_0$, $y_2(\tau + \sigma) = 0$. Similarly, if $y(t) = y_2(t)$ there exists a generalized control $\mu_2 \in M_{\tau,\sigma}(\delta V)$ such that when $y(t) = y_2(t)$ system (1.8) has a solution $y_3(t) \in B_\delta[0]$, $t \in T$ and $y_3(\tau) = x_0$, $y_3(\tau + \sigma) = 0$. Continuing this procedure, we finally obtain a sequence of absolutely continuous functions $\{y_j\}_{j=1}^\infty$ and a sequence of generalized controls $\{\mu_j\}_{j=1}^\infty \subset M_{\tau,\sigma}(\delta V)$ such that $y_j(\tau) = x_0$, $y_j(\tau + \sigma) = 0$, $y_j(t) \in B_\delta[0]$, $t \in T$, $j = 1, 2, \ldots$ and

$$y_{j+1}(t) = \Phi(t, \tau)\left[x_0 + \int_\tau^t \langle \mu_j(s), \Phi(\tau, s)(u + g(s, y_j(s), u))\rangle\, ds\right] \tag{2.1}$$

Moreover, the sequence of functions $\{y_j\}_{j=1}^\infty$ is equicontinuous. Consequently, by the

Arzela-Ascoli theorem /5, p.236/ one can extract from $\{y_j\}_{j=1}^{\infty}$ a subsequence $\{y_{j_k}\}_{k=1}^{\infty}$ which converges uniformly to a function $z \in C(T, B_\delta[0])$. Clearly, $z(\tau) = x_0, z(\tau+\sigma) = 0$. But $M_{\tau,\sigma}(\delta V)$ is a compact subset of the space $(N_T, \|\cdot\|_w)$. Hence we can extract from $\{\mu_{j_k}\}_{k=1}^{\infty}$ a subsequence which converges (in the weak norm $\|\cdot\|_w$) to a generalized control $\nu \in M_{\tau,\sigma}(\delta V)$.

Let us assume that $\mu_{j_k} \to \nu$ as $k \to \infty$. This means that for any function $\varphi \in L_T$

$$\lim_{k\to\infty} \int_T \langle \mu_{j_k}(s), \varphi(s,u)\rangle\, ds = \int_T \langle \nu(s), \varphi(s,u)\rangle\, ds$$

(here see (1.1)), but since for every $t \in T$ the map $(s,u) \to \chi_{[\tau,t]}(s)\,\Psi(\tau, s, u)$, where

$$\Psi(\tau, s, u) \doteq \Phi(\tau, s)(u + g(s, z(s), u)) \tag{2.2}$$

and $\chi_{[\tau,t]}(\cdot)$ is the characteristic function of $[\tau, t]$, is a member of L_T, it follows that

$$\lim_{k\to\infty} \int_T \langle \mu_{j_k}(s), \chi_{[\tau,t]}(s)\,\Psi(\tau,s,u)\rangle\, ds = \int_T \langle \nu(s), \chi_{[\tau,t]}(s)\,\Psi(\tau,s,u)\rangle\, ds$$

This limit relation, together with (2.1), implies that

$$z(t) = \Phi(t,\tau)\Big[x_0 + \int_\tau^t \langle \nu(s), \Psi(\tau,s,u)\rangle\, ds\Big]$$

for all $t \in T \doteq [\tau, \tau+\sigma]$, and $z(\tau) = x_0$, $z(\tau+\sigma) = 0$.

Thus, for any x_0 which satisfies condition (1.9), there exists a generalized control $\nu \in M_{\tau,\sigma}(\delta V) \subset M_{\tau,\sigma}(V)$ such that system (1.4) with $\mu(t) = \nu(t)$ has a solution $z(t)$ for which $z(\tau) = x_0, z(\tau+\sigma) = 0$, i.e., for all $\tau \geqslant 0$ the sphere $B_{\varepsilon\gamma_1/2}[0]$ is contained in the controllable set of system (1.4) in $[\tau, \tau+\sigma]$.

Let $K(t)$ be the closure of the conical hull of $V(t)$.

Corollary 2.1. Suppose that the system

$$x^{\cdot} = A(t)x + v,\ v \in K(t) \tag{2.3}$$

is ULC; let $\varepsilon, \sigma > 0$ be the numbers figuring in the definition of ULC for the system, and assume that $g(t,x,u)$ satisfies conditions 1, 2. Then, if one of conditions (1.7) holds, system (1.4) is ULC.

The proof follows from Theorem 2.1 and the results of /6/.

It should be mentioned that in some cases the question of whether system (2.3) is ULC is easier to handle than in the case of system (1.6), since the structure of the set of admissible controls becomes simpler*. (*For example see TONKOV E.L. and IVANOV A.G., Uniform local controllability in the critical case and questions of oscillation. Preprint FTI Ural. Otd. Akad. Nauk SSSR, Sverdlovsk, 1986.)

Now, since $I(V) \subset M(V)$ ($I(V)$ is identified with $M^1(V)$), it follows that whenever system (1.2) is ULC, the same is true of system (1.4). But if for all $\tau \geqslant 0$ the sphere $B_\varepsilon[0]$ is contained in the controllable set of system (1.4) in $[\tau, \tau+\sigma]$, then one cannot generally state, without further assumptions (e.g., that the velocity field is convex /7, 8/), that $B_\varepsilon[0]$ lies in the controllable set of system (1.2) in $[\tau, \tau+\sigma]$. Nevertheless, the following theorem is true.

Theorem 2.2. Under the assumptions of Theorem 2.1, let the function $g(t,x,u)$ satisfy the following additional conditions:

3) there exists a function $\varkappa_1 \in X^{(1,1)}$ such that for all $x_1, x_2 \in B_\delta[0]$ (concerning the constant $\delta > 0$, as well as the other constants $\varepsilon, \sigma, \gamma_1$ mentioned below, see Lemma 1.1 and Theorem 2.1) and $u \in U$ $|g(t,x_1,u) - g(t,x_2,u)| \leqslant \varkappa_1(t)|x_1 - x_2|$. Then any point x_0 satisfying (1.9) for any $\tau \geqslant 0$ can be steered to any preassigned neighbourhood of zero in $[\tau, \tau+\sigma]$ by means of controls from $I(V)$.

Proof. By Theorem 2.1, for any $\tau \geqslant 0$, x_0 which satisfy (1.9), there exists a generalized control $\nu \in M_{\tau,\sigma}(V)$ such that system (1.4) with $\mu(t) = \nu(t)$ has a solution $z(t) \in B_\delta[0]$, $t \in T \doteq [\tau, \tau+\sigma]$ and $z(\tau) = x_0, z(\tau+\sigma) = 0$. By the approximation lemma /2, 7/, there exists a sequence of functions $\{u_j\}_{j=1}^{\infty} \subset I(V)$, such that $\delta_{u_j} \to \nu$ as $j \to \infty$. Now, if $x_j(t)$ is a solution of system (1.4) with initial condition $x_j(\tau) = x_0$, corresponding to the control $\mu(t) = \delta_{u_j(t)}$, then

$$x_j(t) = \Phi(t,\tau)\Big[x_0 + \int_\tau^t \langle \delta_{u_j(s)}, \Phi(\tau,s)(u + g(s, x_j(s), u))\rangle ds\Big]$$

Using the Gronwall-Bellman inequality, we get

$$\max_{t\in T} |x_j(t) - z(t)| \leqslant \exp\Big(e^{2c\sigma} \sup_{\tau\geqslant 0} \int_\tau^{\tau+\sigma} \varkappa_1(s)\, ds\Big) e^{2c\sigma}\vartheta_j \tag{2.4}$$

$$\vartheta_j \doteq \max_{t\in T} \Big| \int_\tau^t \langle \nu(s) - \delta_{u_j(s)}, \Psi(\tau, s, u)\rangle ds \Big|$$

(the map $\Psi(\tau, s, u)$ is defined by formula (2.2)).

It follows from the conditions imposed on the function $g(t, x, u)$ that the map $(s, u) \to \Psi(\tau, s, u)$ lies in L_T. Hence it follows from the condition: $\delta_{u_j} \to \nu$ as $j \to \infty$ that $\lim \vartheta_j = 0$ as $j \to \infty$; together with (2.4), this proves Theorem 2.2.

3. We will now establish a sufficient condition for the uniform local controllability of a system

$$x^\cdot = \langle \mu(t), f(t, x, u)\rangle,\ x \in \mathbb{R}^n,\ \mu \in M(\Omega),\ \Omega \in \operatorname{comp}(\mathbb{R}^n) \tag{3.1}$$

where it is not assumed that f is differentiable with respect to u (i.e., system (3.1) cannot be expressed in the form (1.4)), and answer certain questions posed in /1/.

The function $f\colon \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is assumed to satisfy the following conditions:

4) the map $t \to f(t, x, u), (x, u) \in \mathbb{R}^n \times \Omega$ is measurable, $f(t, 0, 0) \equiv 0$ and for any compact set $K \subset \mathbb{R}^n$ there exists a function $\Psi_K \in X^{(1,1)}$ such that

$$\max\{|f(t, x, u)|,\ (x, u) \in K \times \Omega\} \leqslant \Psi_K(t)$$

5) the map $x \to f(t, x, u), (t, u) \in \mathbb{R} \times \Omega$ is differentiable; the maps $u \to f(t, 0, u), u \to f_x'(t, 0, u), t \in \mathbb{R}$ are such that for any $\eta > 0$ there exists $\Delta > 0$ for which

$$\sup_{t\geqslant 0} \int_t^{t+1} (|f(s, 0, u)| + |f_x'(s, 0, u) - f_x'(s, 0, 0)|)\, ds < \eta,\ u \in B_\Delta(0)$$

Consider the system

$$x^\cdot = \langle \mu(t)\rangle, f_x'(t, 0, u)\rangle x + \langle \mu(t), f(t, 0, u) + r(t, x, u)\rangle \tag{3.2}$$

with $\mu \in M(\Omega)$, where the function $r\colon \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ satisfies the following conditions:

6) the map $t \to r(t, x, u), (x, u) \in \mathbb{R}^n \times \Omega$ is measurable, $r(t, \cdot, \cdot) \in C(\mathbb{R}^n \times \Omega), t \in \mathbb{R}$ and for any compact set $K \subset \mathbb{R}^n$ there exists a function $\varkappa_K \in X^{(1,1)}$ such that

$$\max\{|r(t, x, u)|,\ (x, u) \in K \times \Omega\} \leqslant \varkappa_K(t)$$

7) there exist a function $a \in X^{(1,1)}$ and a constant $\alpha > 0$ such that

$$|r(t, x, u)| \leqslant a(t)|x|^\alpha,\ \ \forall (t, x, u) \in \mathbb{R} \times B_\gamma[0] \times \Omega\ (\gamma > 0)$$

We shall also assume that system (3.2) has the right uniqueness property.

Let

$$c \doteq \sup_{\tau\geqslant 0} \int_\tau^{\tau+1} |f_x'(t, 0, \cdot)|_{\max}\, dt$$

and choose $\delta \in (0, \gamma]$ so that

$$\sigma e^{2c\sigma} \sup_{\tau\geqslant 0} \int_\tau^{\tau+\sigma} \max_{u\in\Omega_\delta} |f(t, 0, u)|\, dt < \frac{\gamma}{3},\ \Omega_\delta \doteq \Omega \cap B_\delta[0] \tag{3.3}$$

Clearly, if system (3.2) is ULC for $\mu \in M(\Omega_\delta)$, then it is also ULC for $\mu \in M(\Omega)$.

Theorem 3.1. Let $f\colon \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, $r\colon \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ be functions satisfying conditions 4, 5 and 6, 7, respectively. Then system (3.2) with $\mu \in M(\Omega_\delta)$ (consequently, also with $\mu \in M(\Omega)$) is ULC if it is ULC for $r \equiv 0$ and $\mu \in M(\Omega_\delta)$.

Theorem 3.1 answers certain questions posed in /1/. It should also be noted that it is useful in studying controllability questions for system (3.1) when $f_x'(t,0,0)\equiv 0$ and f is not differentiable with respect to u.

To prove the theorem, we need a few facts and some notation.

Let $\Phi(t, s; \mu)$ be the Cauchy operator of the system

$$x^{\cdot} = \langle \mu(t), f_x'(t,0,u)\rangle x \doteq \Big(\int\limits_{\Omega_\delta} f_x'(t,0,u)\,\mu(t)(du)\Big)\, x$$

One verifies directly that

$$D_\tau(\sigma) \doteq \Big\{-\int\limits_\tau^{\tau+\sigma} \langle \mu(t), \Phi(\tau,t;\mu) f(t,0,u)\rangle\, dt,\ \mu \in M_{\tau,\sigma}(\Omega_\delta)\Big\} \tag{3.4}$$

is the controllable set of system (3.2) with $r\equiv 0$, $\mu \in M_{\tau,\sigma}(\Omega_\delta)$, and $D_\tau(\sigma_1)\subset D_\tau(\sigma_2)$, if $\sigma_1 < \sigma_2$. We may therefore assume that $\sigma > 1$.

Lemma 3.1. For any $\tau \geqslant 0$ $D_\tau(\sigma) \in \operatorname{conv}(\mathbb{R}^n)$.

Proof. Using the condition $\mu(t) \in \operatorname{rpm}(\Omega_\delta)$, one can shown that

$$\max_{\tau\leqslant t\leqslant \tau+\sigma} \Phi(\tau,\ t;\ \mu) \,| \leqslant e^{2c\sigma}$$

for all $\tau \geqslant 0$. Hence it follows that the set

$$\{\langle \mu(t), \Phi(\tau,t;\mu) f(t,0,u)\rangle,\ \mu \in M_{\tau,\sigma}(\Omega_\delta)\},\ t\in T \doteq [\tau, \tau+\sigma]$$

is bounded and therefore, by a theorem of Lyapunov /3, p.350/, $D_\tau(\sigma)$ is a bounded convex subset of $\mathbb{R}^n$.

We assert that it is also closed. Let $\{x_j\}_{j=1}^\infty \subset D_\tau(\sigma)$ and suppose that $x_j \to x_0$ as $j\to\infty$. Then (see (3.4))

$$x_j = -\int\limits_T \langle \mu_j(s), \Phi(\tau,s;\mu_j) f(s,0,u)\rangle\, ds$$

Since $\{\mu_j\}_{j=1}^\infty \subset M_{\tau,\sigma}(\Omega_\delta)$, and $M_{\tau,\sigma}(\Omega_\delta)$ is a convex compact subset of the space $(N_T, \|\cdot\|_w)$, we may assume that $\lim\limits_{j\to\infty}\|\mu_j-\mu_0\|_w = 0$ and $\mu_0 \in M_{\tau,\sigma}(\Omega_\delta)$. Hence,

$$\lim_{j\to\infty}\int\limits_T \langle \mu_j(s), f_x'(s,0,u)\rangle\, ds = \int\limits_T \langle \mu_0(s), f_x'(s,0,u)\rangle\, ds$$

which in turn implies that the sequence $\{\Phi(\tau, t;\ \mu_j)\}_{j=1}^\infty$ converges uniformly to $\Phi(\tau,t;\mu_0)$ as $j\to\infty$. Therefore,

$$x_0 = -\int\limits_T \langle \mu_0(s), \Phi(\tau,s;\mu_0) f(s,0,u)\rangle\, ds$$

i.e., $x_0 \in D_\tau(\sigma)$. This proves Lemma 3.1.

Lemma 3.2. Suppose that for $r\equiv 0$, $\mu \in M(\Omega_\delta)$ system (3.2) is ULC, i.e., for all $\tau \geqslant 0$ and some ε, $\sigma > 0$ $B_\varepsilon[0] \subset D_\tau(\sigma)$, the constant $\delta > 0$ is defined by (3.3) and

$$\sup_{\tau\geqslant 0}\int\limits_\tau^{\tau+\sigma} a(s)\,ds < \frac{\varepsilon_0}{2\gamma^\alpha} e^{-2c\sigma},\ \varepsilon_0 \in \Big(0, \frac{2}{3}\gamma e^{-2c\sigma}\Big)$$

Then for any function $y \in C(\mathbb{R}, B_\gamma[0])$ the system

$$x^{\cdot} = \langle \mu(t), f_x'(t,0,u)\rangle x + \langle \mu(t), f(t,0,u) + r(t, y(t), u)\rangle \tag{3.5}$$

where $\mu \in M(\Omega_\delta)$, is ULC, and moreover for any $\tau \geqslant 0$, $x_0 \in B_{\varepsilon_0/2}[0]$ there exists a generalized control $\nu \in M_{\tau,\sigma}(\Omega_\delta)$ such that if $\mu(t) = \nu(t)$ system (3.5) has a solution $x(t) \in B_\gamma[0]$, $t \in [\tau, \tau+\sigma]$ and $x(\tau) = x_0$, $x(\tau+\sigma) = 0$.

Lemma 3.2 is proved with the help of Lemma 3.1, in a manner similar to the proof of Lemma 1.1.

Now, using the scheme of proof of Theorem 2.1. and Lemma 3.2, one can prove Theorem 3.1.

4. In recent years generalized controls have been intensively utilized both in optimal control problems and in game-theoretic situations (see, e.g., /2, 7-11/ and the references cited therein)*. (*See also CHENTSOV A.G., Optimization under conditions of fuzzy constraints. Preprint, Inst. Mat. Mekh., Akad. Nauk SSSR, Sverdlovsk, 1986.) In this paper the apparatus of generalized controls has been used to investigate uniform local controllability for a non-linear system. Our results are important for studies of stability and well-posedness in "main-line processes" (MP).

A brief explanation is in order. In the simplest situation, a MP $(z(\cdot), w(\cdot))$ is a solution of the problem

$$I(x(\cdot), u(\cdot)) \to \min \tag{4.1}$$
$$x^\cdot = f_0(x, u), \quad x \in \mathbb{R}^n, \quad u \in U, \quad U \in \operatorname{comp}(\mathbb{R}^m)$$
$$(x(\cdot), u(\cdot)) \in D$$

where D is some set of functions, specifying the boundary conditions (for example, D might be a set of periodic or almost-periodic functions), and I a functional defined on D. Let us say that the process $(z(\cdot), w(\cdot))$ is uniformly locally stable (or that the problem of the MP is well-posed) if there exist $\varepsilon, \sigma > 0$ such that for any $\tau \geqslant 0$ and any $x_0 \in B_\varepsilon[0]$ there is a control

$$u_0 : [\tau, \tau + \sigma] \to \Omega(t) \doteq U - w(t)$$

such that the system

$$x^\cdot = f_0(z(t) + x, \; w(t) + u) - z^\cdot(t) \tag{4.2}$$

with $u = u_0(t)$, has a solution $x(t)$ which satisfies the conditions $x(\tau) = x_0, x(\tau + \sigma) = 0$ (i.e., the disturbed motion returns to the main line).

This problem possesses several features that make it difficult to handle. First, a MP is defined on the entire real line, so it is important for the local controllability property to be uniform in the time t. Second, since $w(t)$ is optimal, one has the condition $0 \in \partial\Omega(t)$ for all t (i.e., one has the so-called critical case); and, finally, system (4.2) is time-dependent.

Having rewritten system (4.2) in the form of (1.2) with

$$A(t) \doteq f_{0x}'(z(t), w(t)), \quad V(t) \doteq f_{0u}'(z(t), w(t))\,\Omega(t)$$

one can apply the main theorem of /4/. But under the conditons cited in that paper, apart from the assumptions of Theorem 2.1 one also demands that the map $y(\cdot) \to T(\tau, x; y(\cdot))$ be continuous for every fixed pair (τ, x), where $T(\tau, x; y(\cdot))$ is the time needed by the system $x^\cdot = A(t)x + u + g(t, y(t), u)$ to get to zero from the position (τ, x). As shown by Theorem 2.1, this condition is superfluous in the class of generalized controls.

Next, if the function f_0 is not differentiable with respect to u, then Theorem 3.1 can be returned to the main line.

Finally, as can be shown by examples, a solution of problem (4.1) may not exist in D. nevertheless, for natural assumptions on f_0, the "convexified" problem for (4.1) has a solution $(z(\cdot), \nu(\cdot))$ which is a generalized MP. In that case one can use Theorem 2.2.

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